MATHEMATICS LEARNING AREA

YEAR 12 MATHEMATICS METHODS UNIT 3

Assessment type: Response

TASK 3 - TEST 2

CALCULATOR-FREE

Syllabus content : identify anti-differentiation as the reverse of differentiation (3.2.1-3.2.9)

Student Name: _

TIME ALLOWED FOR THIS PAPER

Suggested:
Reading and Working time for Cal Free paper: 25 minutes in class under test conditions

MATERIAL REQUIRED / RECOMMENDED FOR THIS PAPER

TO BE PROVIDED BY THE SUPERVISOR
Question/answer booklet

TO BE PROVIDED BY THE CANDIDATE
Standard Items: pens, pencils, pencil sharpener, highlighter, eraser, ruler, drawing templates, Calculator

IMPORTANT NOTE TO CANDIDATES

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be attempted	Suggested working time (minutes)	Marks available
Calculator Free	3	3	25	**40**
			Marks available:	**/40**
			Task Weighting	7% for the pair of units

Instructions to candidates

Question 1 **(13 marks)**

Consider $\frac{dy}{dx} = f(x)$, this is called a differential equation, more specifically a first order differential equation. To solve or find the general solution of a differential equation is to do the reverse of differentiation.
To find the general solution of a differential equation, then some additional information must be required.

Solve the following differential equations to find the general solution or particular solution.

a) $\frac{dy}{dx} = 3x^2 + (x+1)(x^2+2x+1)^2 + e$ **(3 marks)**

Solution

To solve $\frac{dy}{dx}$, must integrate for both sides

$\int 3x^2 + (x+1)\left(x^2+2x+1\right)^2 + e\,dx$ (1 mark)

$\rightarrow x^3 + \frac{1}{2}\int (2x+2)\left(x^2+2x+1\right)^2 dx + ex + c$ (1 mark)

$\therefore x^3 + \frac{1}{6}(x^2+2x+1)^2 + ex + c$ (1 mark)

Or can also accept

$$\therefore x^3 + \frac{1}{6}(x+1)^4 + ex + c$$

b) $\frac{dy}{dx} = \frac{13x}{\sqrt{4x^2-7}}\ at\ (2, \frac{3}{4})$ **(4 marks)**

Solution

$\int 13x(4x^2-7)^{\frac{1}{2}}\,dx \rightarrow \frac{13}{8}\int 8x\,(4x^2-7)^{\frac{1}{2}}\,dx$ (1 mark)

$\rightarrow y = \frac{13}{12}(4x^2-7)^{\frac{3}{2}} + c$ (1 mark)

$\frac{3}{4} = \frac{13}{12}(4(2)^2-7)^{\frac{3}{2}} + c \rightarrow c = -\frac{57}{2}$ (1 mark)

$\therefore y = \frac{13}{12}(4x^2-7)^{\frac{3}{2}} - \frac{57}{2}$ (1 mark)

c) $\frac{dx}{dy} = \frac{3}{(\tan^2 x + 1 + e^{4x} + \frac{3}{\sqrt[3]{3x+4}})}$ $(Hint: manipulate\ the\ Pythagorean\ identity)$ **(6 marks)**

Solution

$$\frac{dx}{dy} = \frac{1}{(\frac{dy}{dx})}$$ (1 mark)

(Note: this is not the actual explanation to this, as these are limits not fractions, for the actual proof refer to the Sadler Methods Unit 4 Book)

$$\frac{dx}{dy} = \frac{3}{(\tan^2 x + 1 + e^{4x} + \frac{3}{\sqrt[3]{3x+4}})}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\tan^2 x + 1 + e^{4x} + \frac{3}{\sqrt[3]{3x+4}})}{3}$$ (1 mark)

Consider the Pythagorean identity in the formula sheet $\sin^2 x + \cos^2 x = 1$

$$\sin^2 x + \cos^2 x = 1$$

$$\to \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$ (1 mark)

$$\therefore \tan^2 x + 1 = \cos^{-2} x$$ (1 mark)

Thus, by using this identity, use it within $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\left(\cos^{-2} x + e^{4x} + \frac{3}{\sqrt[3]{3x+4}}\right)}{3}$$

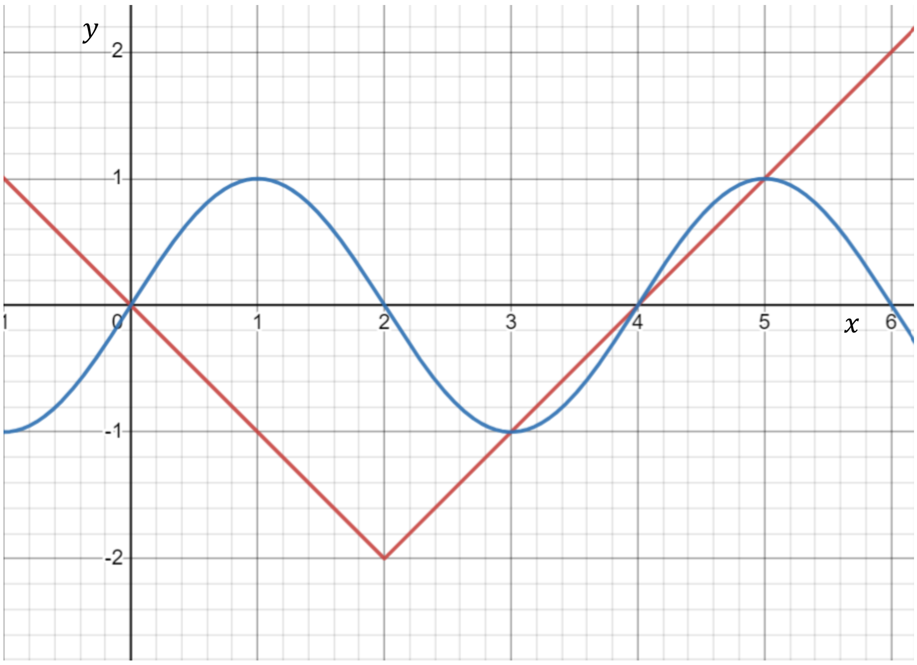
$$\to \frac{dy}{dx} = \frac{1}{3}\cos^{-2} x + \frac{1}{3}e^{4x} + 3(3x+4)^{\frac{1}{3}}$$ (1 mark)

$$\to \int \frac{dy}{dx}\ dx = \int \frac{1}{3}\cos^{-2} x + \frac{1}{3}e^{4x} + 3(3x+4)^{\frac{1}{3}}\ dx$$

Hence, $\therefore y = \frac{1}{3}\tan x + \frac{1}{12}e^{4x} + \frac{3}{4}(3x+4)^{\frac{4}{3}} + c$ (1 mark)

Question 2

Determine the exact area of the region trapped by $y = \sin\frac{1}{2}\pi x$ and $y = |x-2| - 2$, from the origin to $x = 3$. **(8 marks)**



Solution (1st Part)

Equation at the left side of $|x-2| - 2$ of its critical point (at $x = 2$), is $y = -x$
(1 mark)
Equation at the right of $|x-2| - 2$, is $y = x - 4$
(1 mark)
(Note: Can be done by inspection or can accepted in any other method)

Hence can be written as piecewise function

$$\therefore y = \begin{cases} -x \ for\ x \leq 2 \\ x - 4\ for\ x > 2 \end{cases}$$

Solution (2nd Part)

Hence, let A_1 be the area of the first portion from $x = 0$ to $x = 2$

$$A_1 = \int_0^2 \left(\sin\left(\frac{\pi x}{2}\right)\right) - (-x)\ dx \qquad \text{(1 mark)}$$

$$\rightarrow A_1 = \left[\frac{x^2}{2} - \frac{2}{\pi}\cos\left(\frac{x\pi}{2}\right)\right]_0^2$$

$$\therefore A_1 = \left(\frac{4}{\pi} + 2\right) units^2 \qquad \text{(1 mark)}$$

Let A_2 be the area from $x = 2$ to $x = 3$

$$A_2 = \int_2^3 \left(\sin\left(\frac{\pi x}{2}\right)\right) - (x - 4)\ dx \qquad \text{(1 mark)}$$

$$\rightarrow A_2 = \left[-\frac{x^2}{2} - \frac{2}{\pi}\cos\left(\frac{x\pi}{2}\right) + 4x\right]_2^3$$

$$\therefore A_2 = \left(\frac{3}{2} - \frac{2}{\pi}\right) units^2 \qquad \text{(1 mark)}$$

Let the total area be represented as A_T

$$A_T = A_1 + A_2$$

$$\rightarrow A_T = \left(\frac{4}{\pi} + 2\right) + \left(\frac{3}{2} - \frac{2}{\pi}\right) \qquad \text{(1 mark)}$$

$$\therefore A_T = \left(\frac{2}{\pi} + \frac{7}{2}\right) units^2 \qquad \text{(1 mark)}$$

Hence the total exact area of the region trapped by the two functions is $\left(\frac{2}{\pi} + \frac{7}{2}\right) units^2$

Question 3 **(12 marks)**

The function is given as $y = \frac{x}{\cos^2 x} + x\sin x$.

a) Find $\frac{d}{dx}(\frac{x}{\cos^2 x} + x\sin x)$ **(2 marks)**

Solution

$$\frac{d}{dx}\left(\frac{x}{\cos^2 x}\right) + \frac{d}{dx}(x\sin x)$$

$$\rightarrow \frac{(1)(\cos^2 x)-(x)(2\sin x\cos x)}{\cos^4 x} + (1)(\sin x) + (x)(\cos x)$$

(2 marks, by use of quotient rule/product rule, doesn't matter)

$$\rightarrow \frac{\cos^2 x - 2x\sin x\cos x}{\cos^4 x} + \sin x + x\cos x$$

$$\rightarrow \frac{\cos^2 x}{\cos^4 x} - \frac{2x\sin x\cos x}{\cos^4 x} + \sin x + x\cos x$$ (1 mark)

$$\rightarrow \frac{1}{\cos^2 x} - \frac{2x}{\cos^2 x}\left(\frac{\sin x}{\cos x}\right) + \sin x + x\cos x$$

$$\therefore \frac{dy}{dx} = \sin x + x\cos x + \frac{2x\tan x}{\cos^2 x} + \frac{1}{\cos^2 x}$$ (1 mark)

$$\therefore \frac{d}{dx}\left(\frac{x}{\cos^2 x} + x\sin x\right) = \sin x + x\cos x + \frac{2x\tan x}{\cos^2 x} + \frac{1}{\cos^2 x}$$

b) Hence, by using $\frac{d}{dx}(x\sin x)$, find $\int \frac{4x\tan x}{\cos^2 x}dx$. **(6 marks)**

Solution

Firstly, find $\frac{d}{dx}(x\sin x)$

$$\frac{d}{dx}(x\sin x) = \sin x + x\cos x$$ (1 mark)

By using the answer we had in a), we integrate both sides.

$$\int \frac{d}{dx}\left(\frac{x}{\cos^2 x} + x\sin x\right)dx = \int \sin x + x\cos x + \frac{2x\tan x}{\cos^2 x} + \frac{1}{\cos^2 x}\, dx$$

$$\rightarrow \frac{x}{\cos^2 x} + x\sin x + c = -\cos x + \int x\cos x\ dx + \int \frac{2x\tan x}{\cos^2 x}\, dx + \tan x$$

(1 mark)

Back to finding $\frac{d}{dx}(x\sin x)$, we integrate both sides

$$\int \frac{d}{dx}(x\sin x)\, dx = \int \sin x + x\cos x\, dx$$

$$\rightarrow x\sin x + c = -\cos x + \int x\cos x\, dx$$

$$\rightarrow \int x\cos x\, dx = x\sin x + \cos x + c$$ (1 mark)

Hence, from this answer

$$\frac{x}{\cos^2 x} + x\sin x + c = -\cos x + (x\sin x + \cos x) + \int \frac{2x\tan x}{\cos^2 x}\, dx + \tan x$$ (1 mark)

$$\rightarrow \frac{x}{\cos^2 x} + c = \int \frac{2x\tan x}{\cos^2 x}\, dx + \tan x$$

$$\rightarrow \int \frac{2x\tan x}{\cos^2 x}\, dx = \frac{x}{\cos^2 x} - \tan x + c$$ (1 mark)

$$2\int \frac{2x\tan x}{\cos^2 x}\, dx = 2\left(\frac{x}{\cos^2 x} - \tan x\right) + c$$

$$\therefore \int \frac{4x\tan x}{\cos^2 x}\, dx = \frac{2x}{\cos^2 x} - 2\tan x + c$$ (1 mark)

c) Hence find the exact area of the function $f(x) = \frac{4x\tan x}{\cos^2 x}$ between the equations $x = 0$ and $x = \frac{\pi}{4}$ **(2 marks)**

Solution

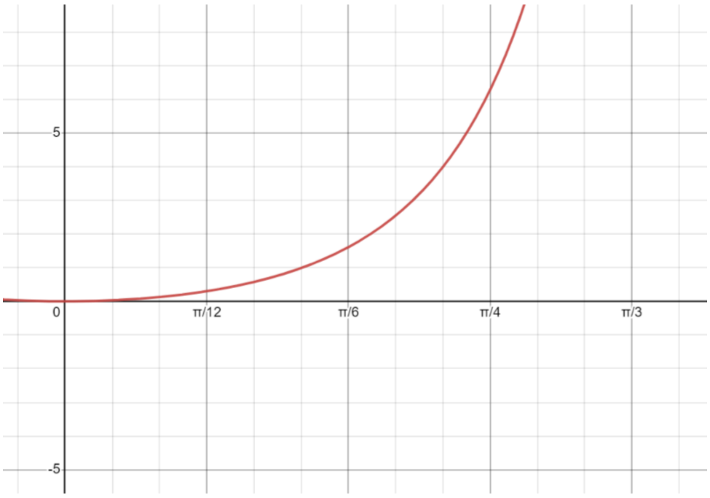
By using the answer in b) and the graph shown at the right

$Area = \int_0^{\frac{\pi}{4}} \left(\frac{4x\tan x}{\cos^2 x}\right) dx$ (1 mark)

$\rightarrow \left[\frac{2x}{\cos^2 x} - 2\tan x\right]_0^{\frac{\pi}{4}}$ (1 mark)

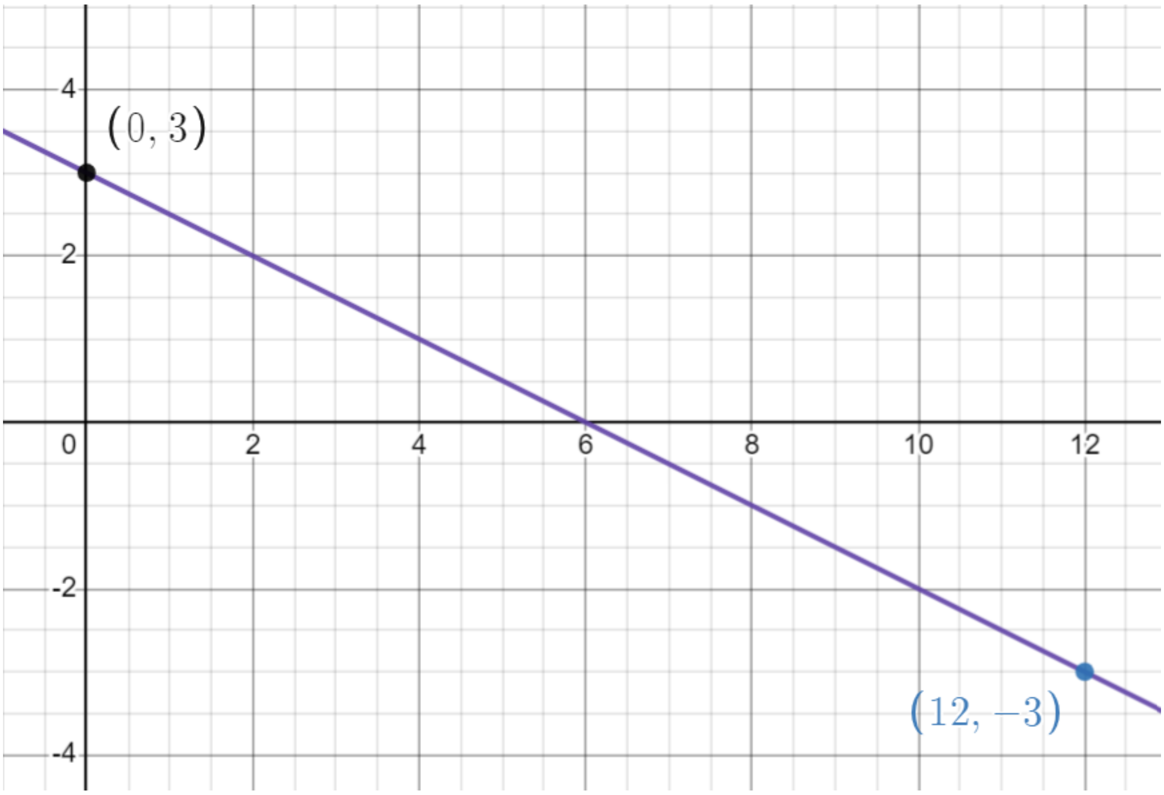
$\therefore Area = (\pi - 2) units^2$

Hence, the exact area of the function between the equations $x = 0$ and $x = \frac{\pi}{4}$ is $(\pi - 2) units^2$



Question 4 **(7 marks)**

Shown below is a graph of $y = f(t)$, and consider only by the interval of $[0,12]$



A man saw this graph and said he wanted to make another equation relating to this, but involving with derivatives and integrals. He came up with the equation of $F(x)$, where…

$$F(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right)$$

Explain by using the knowledge of the fundamental theorem of calculus, why this doesn't change anything. **(2 marks)**

Solution

By *integrating a function $f(t)$ to a definite integral* in respect to any random variable, in this case t. Since it's *upper limit is variable x, deriving it respect to x*, will *revert back to the function in terms of x*.

(1 mark)

Hence this doesn't change anything.

(1 mark)

Or can use an example of any function…

By *integrating a function $f(t)$ to a definite integral* in respect to any random variable, in this case t. Since it's *upper limit is variable x, deriving it respect to x*, will *revert back to the function in terms of x*.

(1 mark)

For example: Let $f(t) = t^2$

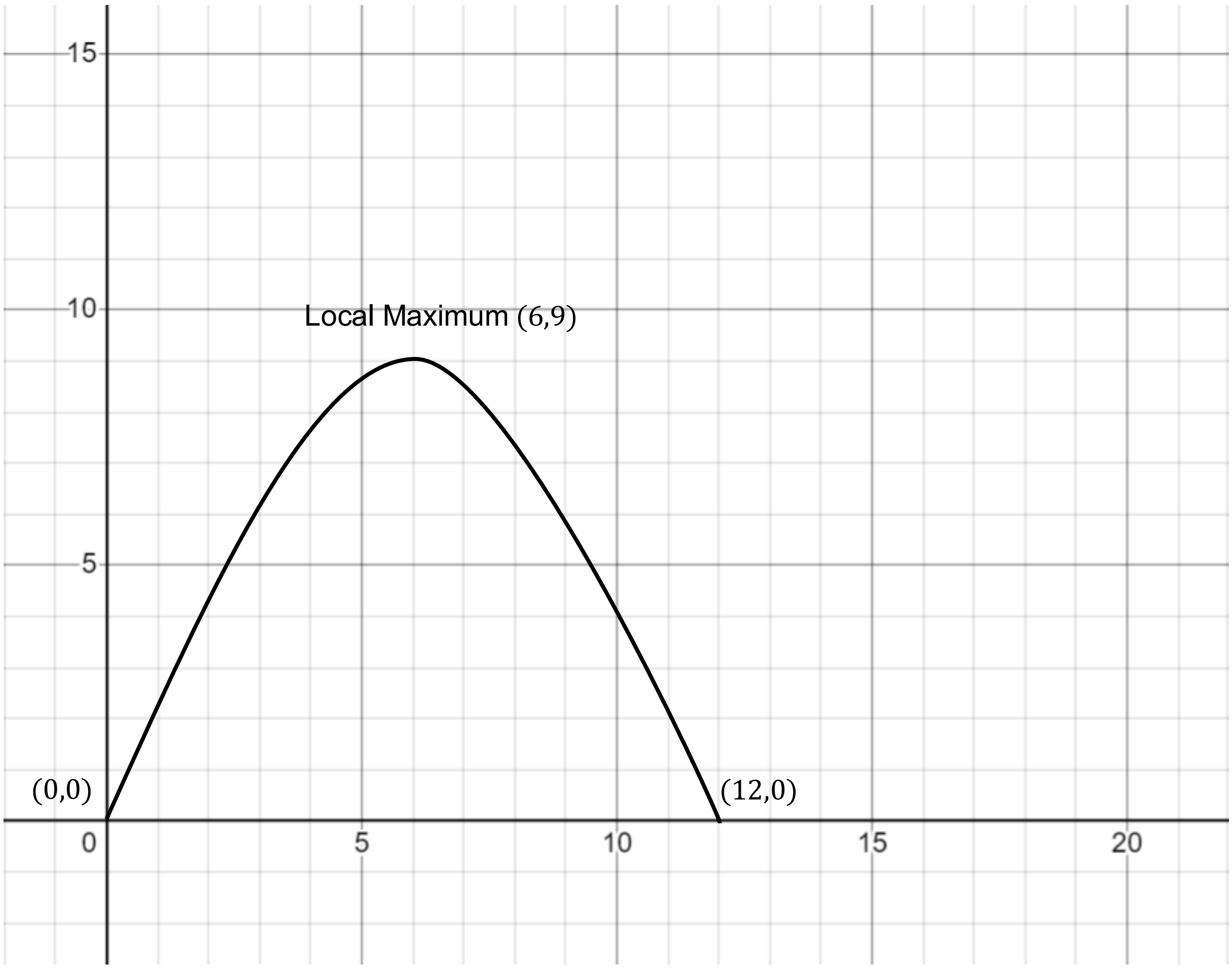
$$\int_0^x t^2\,dt = \frac{x^3}{3} \rightarrow \frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$$

(1 mark)

The man then wanted to make another equation $A(x)$, where…

$$A(x) = \int_0^x f(t)\,dt$$

Sketch $A(x)$ below, labelling the local maximum/minimum, the coordinates at $x = 6$ and $x = 12$ for the given interval $[0,12]$. **(5 marks)**



Solution

Draws correct curve and shape (1 mark)
Labels point $(0,0)$ (1 mark)
Labels point $(12,0)$ (1 mark)

Labels $(6,9)$, and is a Local Maximum (1 mark)
Has correct domains along with closed circles (1 mark)

END OF CALCULATOR-FREE

Additional working space

Question number: __________